11/5 Water condensation processes (and evaporation)

↳ isobaric cooling (example: dew formation) ⇒ saturation ~~condens.~~

↳ isobaric, adiabatic evaporation (ex.: rain) ⇒ cooling

↳ adiabatic, isobaric mixing ⇒ saturation ~~(ex~~
(ex. 1: contrails; ex. 2: punch-through clouds)

↳ adiabatic cooling (ex.: rising air parcel) ⇒ saturation
↳ "interlude": parcel modeling

Recall 1st & 2nd law of thermodynamics

$$đq = dU + p\,dV - \sum \mu_i\, dN_i \,; \quad đq = T\,d\eta$$

$$\Rightarrow dU = T\,d\eta - p\,dV + \sum \mu_i\, dN_i$$

$$= T\,d\eta - p\,dV + \mu_d\, dn_d + \mu_v\, dn_v + \mu_l\, dn_l \; (+\mu_i\, dn_i)$$

$$H = U + pV$$

$$\Rightarrow dH = T\,d\eta + V\,dp + \underbrace{\qquad\qquad\qquad\qquad}$$

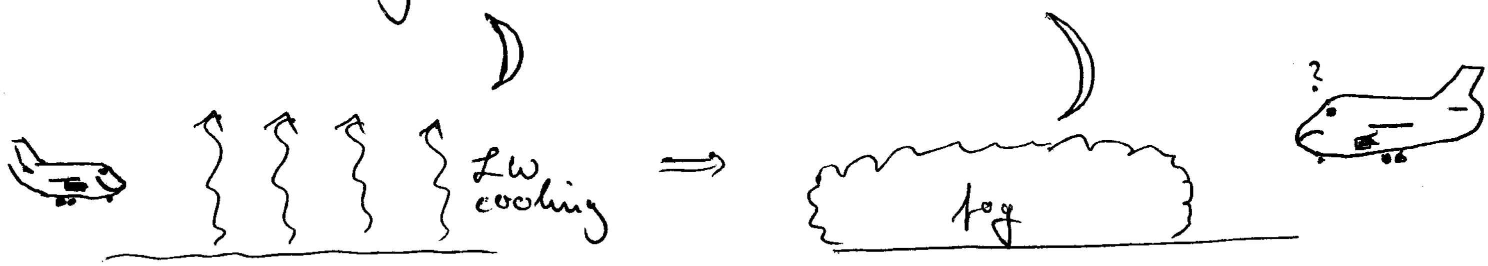
$$G = H - T\eta$$

$$\Rightarrow dG = -\eta\,dT + V\,dp + \underbrace{\qquad\qquad\qquad\qquad}$$

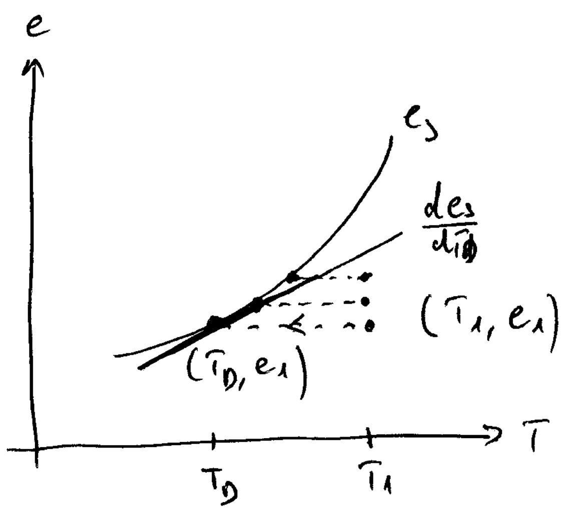
U, H, G are all state functions; it is convenient to use U, H, or G depending on which quantities are conserved in the process under study. For example:

isobaric: $p = \text{const} \Rightarrow V\,dp = 0$
adiabatic: $đq = 0 \Rightarrow T\,d\eta = 0$

Isobaric cooling:



common phenomenon that leads to
- dew (or frost)
- "radiation" fog due to radiative cooling of air over a surface at night



$(T_1, e_1) \longrightarrow e_1 = e_s(T_D)$

$$\frac{d \ln e}{d T_D} = \frac{L}{R_v T_D^2} \quad \text{(because } e_s \text{ follows C-C)}$$

$$\log \frac{e_s}{e} = -\log H = \int_T^{T_D} \frac{L\,(\sim \text{const})}{R\, T_D'^2}\, dT_D' = \frac{L}{R_v} \underbrace{\left(\frac{1}{T_D} - \frac{1}{T}\right)}_{= \frac{T - T_D}{T\,T_D}}$$

$T - T_D$ ← "dewpoint depression"

~~so~~ so T_D is a humidity variable; $T - T_D$ small means air is close to saturation, $T = T_D$ is saturation

Once air is saturated,

$$dq = \underbrace{c_p\, dT}_{\substack{\text{dry air}\\ \text{cooling}}} + \underbrace{L\, dw_v}_{\substack{\text{latent}\\ \text{heat from}\\ \text{condensation}}}$$

$$dw_v = -dw_\ell = dw_s \;; \qquad w_s \approx \varepsilon \frac{e_s}{p} \Rightarrow dw_s = \frac{\varepsilon L e_s}{p R_v T^2}\, dT$$

(because $w_v + w_\ell = \text{const}$)

$$\varepsilon = \frac{M_w}{M_a} = \frac{R_d}{R_v}$$

Cooling by evaporation of water

- adiabatic ($ds = 0$)
- isobaric ($dp = 0$)

$$\Rightarrow dH = \cancel{T\,ds}^{\to 0} + \cancel{V\,dp}^{\to 0} + \cancel{\sum_i \mu_i\, dN_i}^{\to 0 \text{ (phase equilibrium)}}$$

$$= 0 \Rightarrow \text{"isenthalpic" process}$$

now apply this to enthalpy of air, vapor, liquid water in phase equilibrium

$$dH = \frac{\partial H}{\partial T}\,dT + \frac{\partial H}{\partial p}\,dp + \frac{\partial H}{\partial m_d}\,\cancel{dm_d}^{\to 0 \text{ (closed system)}} + \frac{\partial H}{\partial m_l}\,\underbrace{dm_l}_{= -dm_v \text{ (closed system)}} + \frac{\partial H}{\partial m_v}\,dm_v$$

$$= \frac{\partial H}{\partial T}\,dT + \frac{\partial H}{\partial p}\,dp + \underbrace{\left(\frac{\partial H}{\partial m_v} - \frac{\partial H}{\partial m_l}\right)}_{\mathcal{L}_{lv}} dm_v$$

$$= \frac{\partial H}{\partial T}\,dT + \cancel{\frac{\partial H}{\partial p}\,dp}^{\to 0 \text{ for ideal gas}} + \mathcal{L}_{lv}\,dm_v$$

$$\frac{\partial H}{\partial T} = m_d c_{pd} + m_v c_{pv} + m_l c_l$$

~~$m_v \ll m_d$~~ $m_d \gg m_v \gg m_l$

$$\to \frac{\partial H}{\partial T} \approx m_d c_{pd}$$

$$\to dH = m_d c_{pd}\,dT + \mathcal{L}_{lv}\,dm_v$$

$$\Rightarrow dh = \frac{dH}{m_d} = c_{pd}\,dT + \mathcal{L}_{lv}\,dw_v$$

$$\left.\begin{array}{l} c_{pd} \to c_p \\ \mathcal{L}_{lv} \to \mathcal{L} \end{array}\right\} \text{from now on}$$

So for adiabatic, isobaric evaporation of water,

$$dh = c_p \, dT + \underbrace{\mathcal{L} \, dw_v}_{\substack{= -\mathcal{L} \, dw_\ell \quad (w_v + w_\ell = \text{const}) \\ = + \mathcal{L} \, dw_s \quad (H = 1)}} = 0$$

If we evaporate just enough water to achieve saturation (source of water: rain drops, lake, swamp cooler, etc.), then

$$c_p \int_T^{T_w} dT = -\mathcal{L} \int_{w_\ell}^{0} dw_s$$

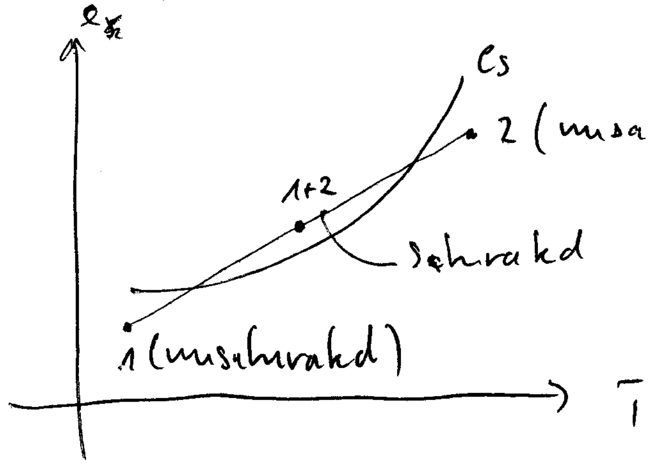
$w_\ell \rightarrow$ amount of liquid that needs to be evaporated to achieve saturation

$$\rightarrow c_p \left(T_w - T \right) = -\mathcal{L} \left(w_s(T_w) - w_\ell \right)$$

"wet bulb temperature"

Why is this useful? T_w can be measured (using a wet-wick thermometer), giving a quick method of determining water vapor mixing ratio

Mixing of airmasses



$$T_{1+2} = \frac{m_1 T_1 + m_2 T_2}{m_1 + m_2} \qquad e_{1+2} = \frac{m_1 e_1 + m_2 e_2}{m_1 + m_2}$$

example: jet exhaust, exhaling on a cold day
(unsaturated airmasses mixing to form a saturated airmass)

How about a saturated airmass mixing with an unsaturated one?

Adiabatic cooling (by lifting of air)

~~Dry air~~ Non-condensing air during ascent:

$$c_p\, dT = v\, dp \qquad v = \frac{RT}{p}$$

$$\to c_p \frac{dT}{T} = R \frac{dp}{p}$$

$$\to c_p \log \frac{T}{T_0} = R \log \frac{p}{p_0}$$

$$\Rightarrow T = T_0 \left(\frac{p}{p_0}\right)^{R/c_p}$$

or $\theta = T \left(\frac{p_0}{p}\right)^{R/c_p}$ = const with p

$\boxed{\text{potential temperature}}$

$$c_p\, dT = -g\, dz \Rightarrow \Gamma_d = -\frac{dT}{dz} = \frac{g}{c_p} = 9.7\ °C/km$$

$\boxed{\text{dry adiabatic lapse rate}}$

Saturation point of moist air: (not saturated yet)

$$H = e/e_s \to \log H = \log e - \log e_s \Rightarrow d\log H = d\log e - d\log e_s$$

$$c_p\, dT = \cancel{dq}^{\,0} + v\, dp = \frac{RT}{p}\, dp$$ (enthalpy in adiabatic ascent)

$$\frac{e}{p} = \text{const} \to d\log e = d\log p$$ (partial pressures)

$$c_p \frac{dT}{T} = R \frac{dp}{p} \Leftrightarrow c_p\, d\log T \overset{(*)}{=} R\, d\log p = R\, d\log e$$

$$d\log e_s = \frac{\varepsilon L}{R_d T}\, d\log T \quad (C\text{-}C) \qquad \left(\varepsilon = \frac{R_d}{R_v}\right)$$

Adiabatic liquid water mixing ratio

Back to (*):

$$c_p\, dT - \mathcal{L}\, dw_\ell = RT\, d\log p = -g\, dz$$

$$\Leftrightarrow dw_\ell = \frac{c_p}{\mathcal{L}}\left(dT + \frac{g}{c_p} dz \right) = \frac{c_p}{\mathcal{L}}\left(\frac{dT}{dz} + \frac{g}{c_p} \right) dz$$

$$\Leftrightarrow \frac{dw_\ell}{dz} = \frac{c_p}{\mathcal{L}} \Big(\underbrace{-\Gamma_s + \Gamma_d}_{} \Big)$$

"adiabatic liquid water ratio"

all the water vapor above saturation at a given height is converted into liquid water —

upper limit of how much liquid water is actually formed in cloud; in practice, entrainment of dry air and rain-out of liquid water reduce w_ℓ.

so $d \log H = \left(\frac{c_p}{R} - \frac{\varepsilon L}{RT} \right) d \log T$

integrate between initial humidity H and saturation to find the saturation temperature:

$$\int_{\log H}^{\log 1} d \log H' = \int_{\log T}^{\log T_s} \left(\frac{c_p}{R} - \frac{\varepsilon L}{RT'} \right) d \log T'$$

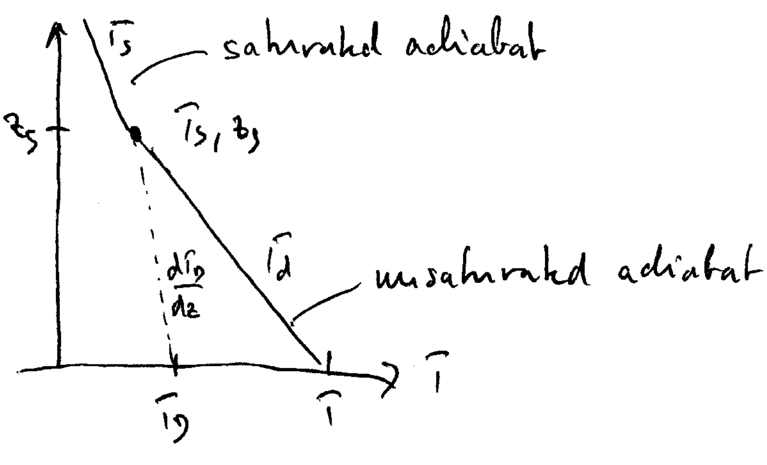
$$\left.\begin{aligned} &\Rightarrow -\log H = \frac{c_p}{R} \log \frac{T_s}{T} + \frac{\varepsilon L}{R} \left(\frac{1}{T_s} - \frac{1}{T} \right) \\ &\text{from } (*) \quad \log \frac{p_s}{p} = \frac{c_p}{R} \log \frac{T_s}{T} \end{aligned}\right\} \quad (T_s, p_s) \text{ is the "saturation point"}$$

Dewpoint-temperature convergence:

$$d \log e = \frac{L}{R_v T_D^2} dT_D \qquad \text{(from earlier)}$$

$$v\,dp = -g\,dz \Rightarrow \frac{RT}{p} dp = -g\,dz \Rightarrow d \overset{d \log e}{\log p} = -\frac{g}{RT} dz$$

$$\Rightarrow \frac{L}{R_v T_D^2} dT_D = -\frac{g}{R_d T} dz \Rightarrow \frac{dT_D}{dz} = -\frac{T_D^2 g}{\varepsilon L T} = -\frac{T_D^2 c_p}{\varepsilon L T} \Gamma_d$$



$z_s \approx 0.12 \, (T_0 - T_{D0}) \, \frac{\text{km}}{\text{K}}$

↑

"lifting condensation level"

Exercise: for the example parcel model run, what is the lifting condensation level (LCL)?

$$\left.\begin{array}{l} w_v = 0.01\ kg/kg \\ T_0 = 289.15\,K \end{array}\right\} T_D = ? \Rightarrow z_s = 0.12\,(T - T_D) = ?$$

$$\cancel{H = \frac{w_v}{w_s} = \frac{10\,g/kg}{10.8\,g/kg} = 93\% \Rightarrow \log H = -0.}$$

$$H = \frac{w_v}{w_s} = \frac{10\,g/kg}{11.5\,g/kg} = 87\% \Rightarrow \log H = -0.141$$

$$-\log H = \frac{L}{R_v}\left(\frac{1}{T_D} - \frac{1}{T}\right) \qquad \frac{L}{R_v} \approx \frac{2.45\cdot 10^6\ J/(kg\,\cancel{K})}{460\ \cancel{10^3}\ J/(kg\,K)}$$

$$\Rightarrow T_D = 287\,K \Rightarrow \cancel{T_D - (}\ T - T_D = 2.24\,K$$

$$\Rightarrow z_s = 270\,m\ ;$$ according to the model, 240m

Saturated adiabatic lapse rate:

$$dH = T\,d\eta + V dp + \sum_i \mu_i\, dn_i$$

capital (extensive)

$$\Rightarrow d\eta = \frac{dH}{T} - \frac{V}{T} dp - \sum_i \frac{\mu_i\, dn_i}{T}$$

lower-case (intensive)

$$\Rightarrow d\eta = \frac{dh}{T} - \frac{v}{T} dp - \sum_i \cancel{\frac{\mu_i\, dw_i}{T}}$$ → 0 in phase equilibrium and conservation of $\sum_i w_i$

recall $dh \approx c_p\, dT + \mathcal{L}\, dw_v$

$$\Rightarrow T\,d\eta = c_p \frac{dT}{\cancel{T}} + \frac{\mathcal{L}}{\cancel{T}} dw_v - \frac{R\cancel{T}T}{p\cancel{T}} dp$$

$$= c_p\, d\log T + \frac{\mathcal{L}}{\cancel{T}} \underset{\underset{-dw_l}{\shortparallel}}{\underset{dw_s}{\underset{\shortparallel}{dw_v}}} - RT\, d\log p = 0 \quad (\text{adiabatic})$$

(*)

recall also $d\log p = -\dfrac{g}{RT} dz$ or $\dfrac{1}{p}\dfrac{dp}{dz} = -\dfrac{g}{RT}$

and $w_s = \dfrac{e_s}{p} \Rightarrow d\log w_s = d\log e_s - d\log p$

$$\Rightarrow \frac{dw_s}{w_s} = \frac{de_s}{e_s} - \frac{dp}{p}$$

$$\text{so } T\,d\eta = 0 = c_p\, dT + g\, dz + \mathcal{L} w_s \left(\frac{de_s}{e_s} - \frac{dp}{p}\right)$$

$$\Leftrightarrow c_p\, dT + g\, dz = -\mathcal{L} w_s \left(\frac{de_s}{e_s} - \frac{dp}{p}\right)$$

$$\Leftrightarrow -\cancel{c_p}\frac{dT}{dz} = \cancel{\frac{1}{c_p}(g}+ \frac{g}{c_p} + \frac{\mathcal{L} w_s}{c_p}\left(\frac{1}{e_s}\frac{de_s}{dz} + \frac{g}{RT}\right)$$

$$\frac{de_s}{dz} = \frac{de_s}{dT}\frac{dT}{dz}$$

$$\frac{de_s}{dT} = \frac{\mathcal{L} e_s}{R_v T^2} \qquad (C\text{-}C)$$

$$\Rightarrow -\frac{dT}{dz} = \frac{g}{c_p} + \frac{\mathcal{L} w_s}{c_p}\left(\frac{\varepsilon \mathcal{L}}{R_d T^2}\frac{dT}{dz} + \frac{g}{\cancel{c_p} R T}\right)$$

$$\rightarrow -\frac{dT}{dz}\overbrace{\left(1 + \frac{\varepsilon \mathcal{L}^2 w_s}{c_p R_d T^2}\right)}^{(2)} = \frac{g}{c_p}\left(1 + \frac{\mathcal{L} w_s}{\cancel{c_p} R_d T}\right) = \Gamma_d \overbrace{\left(1 + \frac{\mathcal{L} w_s}{R T}\right)}^{(1)}$$

$$\Rightarrow \Gamma_s = \Gamma_d \frac{(1)}{(2)}$$

$$\Gamma_s = \Gamma_d \left(\frac{1 + x}{1 + x \cdot \frac{\varepsilon \mathcal{L}}{c_p T}}\right)$$

$$\varepsilon \mathcal{L} \approx 1.5 \cdot 10^6 \text{ J/kg}$$
$$c_p T \approx 3 \cdot 10^5 \text{ J/kg}$$

$\Rightarrow \boxed{\Gamma_s < \Gamma_d}$ $\Rightarrow$ saturated lapse rate is always smaller than dry (why?);

temperature behavior: $w_s \sim \exp\left(-\frac{1}{T}\right)$, so

$$\Gamma_s \sim \Gamma_d \left(\frac{1 + a \exp\left(-\frac{1}{T}\right)/T}{1 + b \exp\left(-\frac{1}{T}\right)/T^2}\right) \rightarrow \Gamma_d \left(\frac{1+0}{1+0}\right) \rightarrow \Gamma_d \text{ at low } T,$$

when saturated air contains very little water vapor that can contribute latent heat.