

1. Consider a planet with an atmosphere in hydrostatic equilibrium. Assume that the atmosphere is an ideal gas. Also assume that the temperature is a maximum at the surface of the planet, and, as height increases, the temperature in the atmosphere decreases linearly (in other words, temperature decreases with height at a constant rate). Derive a formula for atmospheric density as a function of height in this atmosphere.

From the hydrostatic equation for an ideal gas (Eqn. 1.42)

$$\partial p = -\frac{pg}{R_d T} \partial z$$

and a constant lapse rate $\Gamma = -\frac{dT}{dz}$ we get

$$dp = -\frac{pg}{R_d T} dz \left(\frac{-dT/dz}{\Gamma} \right) = -\frac{pg}{\Gamma R_d} \frac{dT}{T}$$

$$\frac{dp}{p} = \left(\frac{g}{\Gamma R_d} \right) \frac{dT}{T}$$

$$\int \frac{dp}{p} = \int \left(\frac{g}{\Gamma R_d} \right) \frac{dT}{T}$$

$$\ln \frac{p}{p_0} = \left(\frac{g}{\Gamma R_d} \right) \ln \frac{T}{T_0}$$

$$p = p_0 \left(\frac{T}{T_0} \right)^{\left(\frac{g}{R_d \Gamma} \right)}$$

which is Eqn 1.48. Then dividing both sides by RT and noting that for an ideal gas

$$\rho = \frac{p}{RT}, \text{ we get}$$

$$\frac{p}{RT} = \rho = \frac{p_0}{RT} \left(\frac{T}{T_0} \right)^{\frac{g}{R_d \Gamma}}$$

2. The saturation vapor pressure at a temperature of 30°C is 42.4 hPa. The gas constant for dry air is 287 J K⁻¹ kg⁻¹. The gas constant for water vapor is 461 J K⁻¹ kg⁻¹. Consider moist air at a temperature of 30°C, a pressure of 1,000 hPa, and a relative humidity of 50%. Find the values of the following quantities:
- vapor pressure: $e = H * e_s = 0.50 * 42.4 = 21.2$ hPa [Eqn. 4.34a]
 - mixing ratio: $w_v = m_v / m_d = (M_v / M_d) * (e / (p - e)) = 0.0135$ [Eqn. 4.36]
 - specific humidity: $q_v = m_v / (m_v + m_d) = w_v / (w_v + 1) = 0.0133$ [Eqn. 1.20]
 - specific heat at constant pressure: $c_p \sim (7R/2)(1 + 0.87q_v) = 1016$ J/K/kg [Eqn. 2.65]
 - virtual temperature: $T_v = (1 + 0.608q_v) * T = 305.6$ K [Eqn. 1.25]

3. In addition to the constants given above, here is one more: the saturation vapor pressure at a temperature of 40°C is 73.8 hPa. Assuming that the latent heat of vaporization is constant, use this information to calculate the numerical value for this latent heat.

The Clausius Clapeyron equation can be integrated if L is assumed constant, and the result is Eqn. 4.23. Using 30°C=303K and 40°C=313K, and knowing saturation vapor pressure values for each, the only unknown is L . Solving Eqn. 4.23,

$$e_2 = e_1 \exp\left[-\frac{L_{lv}}{R_v}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]$$

$$L_{lv} = -R_v\left(\frac{T_1 T_2}{T_1 - T_2}\right)\left(\ln \frac{e_2}{e_1}\right) = -461\left(\frac{303 \times 313}{303 - 313}\right)\left(\ln \frac{73.8}{42.4}\right) = 2.4 \times 10^6$$

$$L = 2.4 \times 10^6 \text{ J/kg.}$$

4. Define the following terms, briefly and clearly, in light of their use in the kinetic theory of gases and the first and second laws of thermodynamics:
- an ideal gas: vapor whose molecules have collisions with perfect elasticity, typical at low pressures (for air ≤ 1 atm) and high temperatures (typically ≥ 300 K); vapor that satisfies $pv=RT$ and has the properties that $dh=c_p dT$, $du=c_v dT$, $c_p - c_v = R$ (p. 44).
 - temperature: the intensive property describing the internal energy of a gas, which for an ideal gas depends only on the average speed of the molecules.
 - entropy: a state property whose differential describes the amount of energy that is not available for doing work for a reversible process (in which maximum work is done) and satisfies the criteria of an exact differential; it can be evaluated from Eqn. 2.25a and 2.26b:

$$d\eta = \left(\frac{dQ}{T}\right)_{rev} = c_p d(\ln T) - R d(\ln p).$$

- d) exact differential: a function ξ for which $d\xi$ has the properties (1) for any closed path $\oint d\xi = 0$, and (2) for $\xi(x,y)$ where x and y are independent, then

$$d\xi = \left(\frac{\partial \xi}{\partial x}\right)_y dx + \left(\frac{\partial \xi}{\partial y}\right)_x dy \equiv M dx + N dy \Rightarrow \left(\frac{\partial M}{\partial y}\right)_x = \left(\frac{\partial N}{\partial x}\right)_y.$$

- e) enthalpy: a state property whose differential describes the change in heat for a constant-pressure process and satisfies the criteria of an exact differential; it is defined as $H=U+pV$ (Eqn. 2.12).
5. It has been estimated from satellite observations that variations in solar radiance during the last 20 years amounted to $\leq 0.2 \text{ W m}^{-2}$, or less than 0.1% of the incoming shortwave radiation. Calculate the approximate change in the temperature at the Earth's surface for a 0.1% decrease in solar luminosity for a simplified climate model. State all assumptions, simplifications, and equations used. Values of constants that you may need are Earth's albedo 0.31, solar luminosity $3.92 \times 10^{26} \text{ W}$, Earth-sun distance $1.50 \times 10^{11} \text{ m}$, Stefan-Boltzmann constant $5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$.

Assume that: (1) the earth behaves as a blackbody, (2) atmosphere is transparent to non-reflected portion of the solar beam; (3) atmosphere in radiative equilibrium with surface; (4) atmosphere absorbs all the infrared emission. Then, at equilibrium, the incoming shortwave flux and outgoing longwave flux are equal (i.e. there is no accumulation) so for the normal solar luminosity we can write:

$$F_L = \sigma T_{\text{surf}}^4 \quad (\text{assumption 1; Eqn. 3.20})$$

$$F_S = F_L \quad (\text{assumption 2-4})$$

$$0.25 * S_0 (1 - \alpha_p) = \sigma T_{\text{surf}}^4 \quad (\text{Eqn. 12.1b})$$

$$T_{\text{surf}} = [0.25 * S_0 (1 - \alpha_p) / \sigma]^{0.25}$$

where $S_0 = L_0 / (4\pi d^2) = 1.3938 \times 10^3 \text{ W m}^{-2}$ (p. 332), $\alpha_p = 0.31$, $\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$

$$T_{\text{surf}} = 254.84 \text{ K}$$

Reducing L to 99.9% of its value, we get $S_0' = 0.999 * 1.3938 \times 10^3 = 1.3924 \times 10^3 \text{ W m}^{-2}$

$$T_{\text{surf}} = 254.78 \text{ K}$$

The resulting cooling of 0.06K associated with a 0.1% reduction in solar radiation is negligible for this simplified climate model.

6. Name the five main components of the atmosphere. (a) If all components are in the gas phase, how many degrees of freedom are there in the system? (b) If water condenses or freezes, does that number increase or decrease? (c) If new components are added by pollution, how does that change (i) the number of possible phases and (ii) the degrees of freedom of the atmosphere?

The five main components of the atmosphere are nitrogen (N_2), oxygen (O_2), carbon dioxide (CO_2), argon (Ar), and water (H_2O).

(a) For this system, we can use the Gibbs phase rule (Eqn. 4.2) with $\chi=5$, $\varphi=1$, $f=\chi-\varphi+2=6$.

(b) Condenses $\varphi=2$, $f=5$ [decrease]; freezes $\varphi=2$, $f=5$ [decrease] (both: $\varphi=3$, $f=4$ [decrease]).

(c) (i) number of phases that can exist at atmospheric pressure may increase with additional components, since multiple liquid and solid phases may form; (ii) degrees of freedom increase with the number of components and will decrease with the number of phases.